

# On exceptional rigid local systems

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## Abstract

We prove new instances of Simpson's rigidity conjecture which states that quasi-unipotent rigid local systems should be motivic. We construct new relative motives over the fourfold punctured Riemann sphere which give rise to  $G_2$ -rigid local systems which are not rigid in the group  $\mathrm{GL}_7$ .

## Introduction

Let  $G$  be a reductive algebraic group over  $\mathbb{C}$  and let  $X$  be a smooth quasi-projective complex variety. Let us call a representation  $\rho : \pi_1(X) \rightarrow G$  to be *G-rigid*, if the set theoretic orbit of the representation  $\rho$  in the representation space  $\mathrm{Hom}(\pi_1(X), G)$  under the action of  $G$  is an open subset.

Note that by fixing an embedding of  $G$  into a general linear group  $\mathrm{GL}_n(\mathbb{C})$ , any representation  $\rho : \pi_1(X) \rightarrow G$  corresponds uniquely to a local system  $\mathcal{L}_\rho$  on  $X$ , see [3]. On the other hand, any local system on  $X$  whose monodromy lies in  $G \leq \mathrm{GL}_n$  gives rise to a representation  $\rho = \rho_\mathcal{L} : \pi_1(X) \rightarrow G$ . The local system  $\mathcal{L} = \mathcal{L}_\rho$  is called *G-rigid*, if  $\rho$  is *G-rigid*. A local system is called *quasi-unipotent*, if the eigenvalues of the local monodromies of  $\mathcal{L}$  are roots of unity.

The following conjecture is motivated by Hodge theory and appears in work of Simpson [10] (for projective  $X$ ):

**Rigidity Conjecture:** Let  $G$  be a reductive complex algebraic group and let  $X$  be a smooth quasi-projective complex variety. Then any quasi-unipotent *G-rigid* local system on  $X$  is *motivic*, i.e., it is a subfactor of a higher direct image sheaf  $R^i \pi_* (\mathbb{C})$ , where  $\pi : Y \rightarrow X$  is some smooth morphism.

The case of  $\mathrm{GL}_n$ -rigidity for representation of  $\pi_1(X)$ , where  $X$  is the punctured Riemann sphere, is probably the most important one. Many classical ordinary

differential equations (e.g., the generalized hypergeometric differential equations) give rise to  $\mathrm{GL}_n$ -rigid local systems on  $\mathbb{P}^1 \setminus \{x_1, \dots, x_r\}$  – and can be studied in detail using the concept of rigidity, see e.g. [2] and [8]. Moreover, by the work of N. Katz [8], there exist a uniform description of all irreducible  $\mathrm{GL}_n$ -rigid local systems (loc. cit., Chap. 6) and the rigidity conjecture has been proven for irreducible  $\mathrm{GL}_n$ -rigid local systems (loc. cit., Chap. 8).

Let  $G \leq H$  denote an embedding of algebraic groups (think about the embedding of the exceptional simple group  $G_2$  into the general linear group  $\mathrm{GL}_7$ ). If  $\mathcal{L}$  is a  $G$ -rigid local system, then in most cases  $\mathcal{L}$  is not  $H$ -rigid. In a recent paper [6], we use Katz' work and certain  $G_2$ -rigid representations of  $\pi_1(\mathbb{P}^1 \setminus \{0, 1, \infty\})$  which are (surprisingly)  $\mathrm{GL}_7$ -rigid, in order to construct motivic  $G_2$  local systems. Unfortunately, most  $G_2$ -rigid local systems are not  $\mathrm{GL}_7$ -rigid – and the rigidity conjecture for them cannot be deduced from Katz' work so easily.

It is the aim of this article to show that nevertheless, some  $G_2$ -rigid local systems are motivic although they are not  $\mathrm{GL}_7$ -rigid. We prove the following result (where  $J(n_1, \dots, n_k)$  denotes a unipotent matrix in Jordan canonical form which decomposes into blocks of length  $n_1, \dots, n_k$  and  $\zeta_3$  denotes a primitive third root of unity):

**Theorem 1:** *Let  $x_1, x_2, x_3$  be pairwise distinct complex numbers. Then the following holds:*

- (i) *There exists a  $G_2(\mathbb{C})$ -rigid local system  $\mathcal{H}$  on  $\mathbb{P}^1 \setminus \{x_1, x_2, x_3, \infty\}$  such that the Jordan canonical forms of the local monodromies at  $x_1, x_2, x_3, \infty$  are as follows:*

$$J(2, 2, 1, 1, 1), \quad J(2, 2, 1, 1, 1), \quad \mathrm{diag}(1, \zeta_3, \zeta_3, \zeta_3, \zeta_3^{-1}, \zeta_3^{-1}, \zeta_3^{-1}), \quad J(3, 3, 1).$$

- (ii) *The Zariski closure of the image of the monodromy of  $\mathcal{H}$  coincides with the exceptional simple group  $G_2(\mathbb{C})$ .*
- (iii) *The local system  $\mathcal{H}$  is  $\mathrm{SO}_7(\mathbb{C})$ -rigid but not  $\mathrm{GL}_7(\mathbb{C})$ -rigid.*
- (iv) *The rigidity conjecture holds for  $\mathcal{H}$ .*

The proof of Theorem 1 is given in Section 2. An outline of the proof is as follows: We start with two  $\mathrm{GL}_2(\mathbb{C})$ -rigid local systems and take an appropriate tensor product of them. This gives rise to a local system of rank four on

$\mathbb{P}^1 \setminus \{x_1, x_2, x_3, \infty\}$  which is  $\mathrm{SO}_4(\mathbb{C})$ -rigid but not  $\mathrm{GL}_4(\mathbb{C})$ -rigid. Then we apply the middle convolution and suitable tensor operations twice end up with the above  $G_2(\mathbb{C})$ -rigid local system  $\mathcal{H}$  (it is constructed in Section 2). The rigidity conjecture for  $\mathcal{H}$  follows then from the motivic interpretation of the middle convolution as given in the work of Katz [8], Chap. 8. It seems remarkable, that the weight of  $\mathcal{H}$  (as a variation of Hodge-structures) is 4 contrary to the  $G_2$ -local system considered in [6] which has weight 6.

This article has grown out from a question of P. Deligne about a possible motivic interpretation of the other  $G_2$ -rigid (but non- $\mathrm{GL}_7$ -rigid) local systems that exist besides the  $G_2$ -rigid local systems considered in [6] and some related  $G_2$ -rigid local systems which were communicated to the authors by P. Deligne and N. Katz. One may hope that the above approach leads to a verification of the rigidity conjecture for rigid local systems with values in other exceptional groups.

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## 1 Preliminaries

**1.1 Rigid local systems and rigid tuples.** Let  $X$  be  $\mathbb{P}^1 \setminus D$ , where  $D = \{x_1, \dots, x_r\} \subseteq \mathbb{P}^1$ . Fix a homotopy base  $\gamma_1, \dots, \gamma_r$  of  $\pi_1(X)$  satisfying the product relation  $\gamma_1 \cdots \gamma_r = 1$ . The category of local systems of  $\mathbb{C}$ -vector spaces will be denoted by  $\mathrm{LS}(X)$ . It is well known that any local system corresponds uniquely to its *monodromy representation*

$$\rho_{\mathcal{L}} : \pi_1(X, x_0) \longrightarrow \mathrm{GL}(\mathcal{L}_{x_0}),$$

see e.g. [3]. Thus any local system  $\mathcal{L} \in \mathrm{LS}(X)$  of rank  $n$  corresponds to its *monodromy tuple*  $\mathbf{g}_{\mathcal{L}} = (g_1, \dots, g_r)$ , where  $g_i \in \mathrm{GL}(\mathcal{L}_{x_0}) \simeq \mathrm{GL}_n(\mathbb{C})$  is the image of  $\gamma_i$  under the monodromy representation  $\rho_{\mathcal{L}}$ . By construction, the product relation  $g_1 \cdots g_r = 1$  holds for any monodromy tuple  $\mathbf{g}_{\mathcal{L}}$ ,  $\mathcal{L} \in \mathrm{LS}(X)$ . (The concept of a monodromy tuple enables in many cases the explicit computation of a local system  $\mathcal{L} \in \mathrm{LS}(X)$ .)

Let  $G \leq \mathrm{GL}_n(\mathbb{C})$  be a reductive complex algebraic group. By definition, a local system  $\mathcal{L} \in \mathrm{LS}(X)$  with monodromy tuple  $\mathbf{g} = (g_1, \dots, g_r) \in G^r$  (resp.  $\rho_{\mathcal{L}}$ ) is *G-rigid*, if there are (up to simultaneous  $G$ -conjugation) only finitely many tuples  $\mathbf{h} = (h_1, \dots, h_r) \in G^r$  with  $h_1 \cdots h_r = 1$  and such that  $h_i$  is  $G$ -conjugate to  $g_i$ .

In this case the tuple  $\mathbf{g}$  is called *G-rigid*.

The following theorem is often useful to detect *G-rigid* tuples:

**1.1.1 Theorem.** *Let  $G \leq \mathrm{GL}_n(\mathbb{C})$  be a reductive algebraic subgroup. Let  $\mathcal{L}$  be an irreducible local system of rank  $n$  whose monodromy tuple  $\mathbf{g} = (g_1, \dots, g_r)$  is contained in  $G^r$ . Then the local system  $\mathcal{L}$  (resp. its monodromy tuple  $\mathbf{g}$ ) is *G-rigid*, if and only if the following dimension formula holds:*

$$\sum_{i=1}^r \mathrm{codim}(C_G(g_i)) = 2(\dim(G) - \dim(Z(G))),$$

where  $C_G(g_i)$  denotes the centralizer of  $g_i$  in  $G$ , the codimension is taken relative to  $G$ , and  $Z(G)$  denotes the centre of  $G$ .

**Proof:** The necessity of the dimension formula for *G-rigidity* is proven in [11]. The other direction follows from the same arguments as in [12], Section 7.  $\square$

**1.2 Operations on local systems** The following operations on local systems on  $X$  will play a role below:

On the one hand side, there is the usual tensor product  $\mathcal{L}_1 \otimes \mathcal{L}_2$  of local systems  $\mathcal{L}_1, \mathcal{L}_2 \in \mathrm{LS}(X)$  having ranks  $n_1, n_2$ , respectively. Let  $\mathbf{g}_{\mathcal{L}_1} = (g_1, \dots, g_r)$  and  $\mathbf{g}_{\mathcal{L}_2} = (h_1, \dots, h_r)$  be the monodromy tuples of  $\mathcal{L}_1$  and  $\mathcal{L}_2$  (resp.). Then

$$\mathbf{g}_{\mathcal{L}_1 \otimes \mathcal{L}_2} = \mathbf{g}_{\mathcal{L}_1} \otimes \mathbf{g}_{\mathcal{L}_2} := (g_1 \otimes h_1, \dots, g_r \otimes h_r) \in \mathrm{GL}_{n_1 n_2}^r,$$

where  $g_i \otimes h_i$  denotes the usual Kronecker product of matrices. Thus, the tensor product of local systems is very well understood from the computational side. Moreover, if  $\mathcal{L}_1$  and  $\mathcal{L}_2$  are irreducible and motivic, then it follows from the Künneth formula that  $\mathcal{L}_1 \otimes \mathcal{L}_2$  is also motivic.

The next operation on local systems on

$$X = \mathbb{A}^1 \setminus \mathbf{x}, \mathbf{x} = \{x_1, \dots, x_r\} = \mathbb{P}^1 \setminus (\mathbf{x} \cup \infty),$$

was introduced by N. Katz in [8] and is much less obvious: Let  $p(x) = \prod_{i=1}^r (x - x_i)$  and let  $L$  be the divisor on  $\mathbb{A}^2 = \mathbb{A}_x^1 \times \mathbb{A}_y^1$  associated to  $p(x) \cdot p(y) \cdot (y - x) = 0$ . Let  $U = \mathbb{A}^2 \setminus L$  and let

$$j : U \longrightarrow \mathbb{P}_{\mathbb{A}_y^1 \setminus \mathbf{x}}^1, (x, y) \longmapsto ([x, 1], y)$$

(by abuse of notation, the divisor on  $\mathbb{A}_y^1$  which is given by  $p(y) = 0$  is also denoted by  $\mathbf{x}$ ). Let further  $\text{pr}_i : U \rightarrow \mathbb{A}^1 \setminus \mathbf{x}$ ,  $i = 1, 2$ , denote the  $i$ -th projection, let

$$d : U \longrightarrow \mathbb{G}_m = \mathbb{A}^1 \setminus \{0\}, (x, y) \longmapsto y - x$$

be the *difference map* and let  $\bar{\text{pr}}_2 : \mathbb{P}_{\mathbb{A}^1 \setminus \mathbf{x}}^1 \rightarrow \mathbb{A}^1 \setminus \mathbf{x}$  be the natural projection. Following Katz [8], Chap. 8, we define a middle convolution operation on  $\text{LS}(\mathbb{A}^1 \setminus \mathbf{x})$  as follows:

**1.2.1 Definition.** For a character  $\chi : \pi_1^{\text{ét}}(\mathbb{G}_m) \rightarrow \mathbb{C}^\times$ , let  $\mathcal{L}_\chi \in \text{LS}(\mathbb{G}_m)$  denote the associated local system. For  $\mathcal{L} \in \text{LS}(\mathbb{A}^1 \setminus \mathbf{x})$ , the local system

$$\text{MC}_\chi(\mathcal{L}) := R^1(\bar{\text{pr}}_2)_*(j_*(\text{pr}_1^*(\mathcal{L}) \otimes d^*(\mathcal{L}_\chi))) \in \text{LS}_R(\mathbb{A}^1 \setminus \mathbf{x})$$

is called the *middle convolution of  $\mathcal{L}$  with  $\mathcal{L}_\chi$* .

**1.2.2 Remark.** The above definition amounts to a special case of the middle convolution of perverse sheaves, introduced in [8]. It follows from [4], L. 3.5.5, that if  $\mathcal{L}$  is irreducible and has at least two nontrivial local monodromies away from  $\infty$ , then the monodromy tuple of  $\text{MC}_\chi(\mathcal{L})$  is given by the tuple  $\text{MC}_\lambda(g_\mathcal{L})$ , where  $\text{MC}_\lambda$  is the tuple-transformation introduced in [5] (the tuple  $\text{MC}_\lambda(\mathbf{g}_\mathcal{L})$  is constructed from  $\mathbf{g}_\mathcal{L}$  by an explicit receipt using only linear algebra). Thus, the middle convolution of local systems with Kummer sheaves is also well understood from the computational side.

The following Proposition can be proved using the the same arguments as [8], Chap. 8:

**1.2.3 Proposition.** Let  $\pi : Y \rightarrow \mathbb{A}^1 \setminus \mathbf{x}$  be a smooth proper morphism and let  $\mathcal{L}$  be an irreducible subsheaf of  $R^i\pi_*(\mathbb{C}) \in \text{LS}(\mathbb{A}^1 \setminus \mathbf{x})$  which has at least two non-trivial local monodromies away from  $\infty$ . Assume further that  $\mathcal{L}$  has weight  $k$  (as variation of Hodge structures). For a natural number  $n$ , let

$$\alpha_n : \mathbb{G}_m \rightarrow \mathbb{G}_m, x \mapsto x^n$$

and let

$$\Pi := \text{pr}_2 \circ (\pi \boxtimes \alpha_n),$$

where  $\pi \boxtimes \alpha_n$  denotes the fibre product of the pullbacks of  $\pi$  and  $\alpha_n$  to  $U$ . Let  $\chi : \pi_1(\mathbb{G}_m) \rightarrow \mathbb{C}$  be the character of  $\pi_1(\mathbb{G}_m)$  which sends a counterclockwise generator of  $\pi_1(\mathbb{G}_m)$  to  $e^{2\pi i/n}$ . Then  $\text{MC}_\chi(\mathcal{L}) \in \text{LS}(\mathbb{A}^1 \setminus \mathbf{x})$  is a subfactor of  $R^1\Pi_*(\mathbb{C}) \in \text{LS}(\mathbb{A}^1 \setminus \mathbf{x})$  and the weight of  $\text{MC}_\chi(\mathcal{L})$  as a variation of Hodge structures is equal to  $k + 1$ .

**1.3 Construction of local systems with finite monodromy** The following construction of local systems with finite monodromy will be used in Section 2 below: Let  $\rho_f : \pi_1^{\text{ét}}(X) \rightarrow G$  be the surjective homomorphism associated to a finite Galois cover of  $X$  with Galois group  $G$  and let  $\alpha : G \rightarrow \text{GL}_n(\mathbb{C})$  be a representation. Then the composition  $\alpha \circ \rho_f : \pi_1^{\text{ét}}(X) \rightarrow \text{GL}_n(\mathbb{C})$  corresponds to a local system with finite monodromy on  $X$  which is denoted by  $\mathcal{L}_{f,\alpha}$ . Note that  $\mathcal{L}_{f,\alpha}$  is motivic by construction.

## 2 The proof of Theorem 1

Let  $x_1, x_2, x_3 \in \mathbb{C}$  be pairwise distinct, let  $X = \mathbb{P}^1 \setminus \{x_1, x_2, x_3, \infty\}$  and let  $\zeta_3$  denote a fixed primitive third root of unity. Fix a homotopy base  $\gamma_i$ ,  $i = 1, 2, 3, 4$  of  $\pi_1(X, x_0)$  such that  $\gamma_i$ ,  $i = 1, 2, 3$ , is a simple loop around the missing point  $x_i$ , such that  $\gamma_4$  is a simple loop around  $\infty$ , and such that  $\gamma_1 \cdots \gamma_4 = 1$ .

**2.1 Construction of the underlying local system** Using the notation of Section 1.3, let  $\mathcal{L}_i = \mathcal{L}_{f_i, \alpha} \in \text{LS}(X)$ ,  $i = 1, 2, 3$ , be as follows:

- The Galois cover  $f_1 : Y_1 \rightarrow X$  is the cover of  $X$  with Galois group

$$Z_3 = \langle \sigma \mid \sigma^3 = 1 \rangle$$

which is ramified at  $x_1, x_3$  and  $\infty$ , and  $\alpha$  sends  $\sigma$  to  $\zeta_3 \in \mathbb{C}^\times = \text{GL}_1(\mathbb{C})$ . Thus the monodromy tuple of  $\mathcal{L}_1 \in \text{LS}(X)$  is equal to  $\mathbf{g}_1 = (\zeta_3, 1, \zeta_3, \zeta_3)$ .

- The Galois cover  $f_2 : Y_2 \rightarrow X$  is the cover of  $X$  with Galois group  $Z_3$  which is ramified at  $x_2, x_3$  and  $\infty$ . Thus the monodromy tuple of  $\mathcal{L}_2 \in \text{LS}(X)$  is equal to  $\mathbf{g}_2 = (1, \zeta_3, \zeta_3, \zeta_3)$ .
- The Galois cover  $f_3 : Y_3 \rightarrow X$  is the cover of  $X$  with Galois group  $Z_3$  which is ramified at  $x_3$  and  $\infty$ . Thus the monodromy tuple of  $\mathcal{L}_3$  is  $\mathbf{g}_3 = (1, 1, \zeta_3, \zeta_3^{-1})$ .

Let  $\chi : \pi_1(\mathbb{G}_m) \rightarrow \mathbb{C}^\times$  be the character which sends a counterclockwise generator of  $\pi_1(\mathbb{G}_m)$  to  $\zeta_3$  and let  $\chi^{-1}$  be the dual character. Consider the following sequence of tensor operations and middle convolutions of local systems:

$$(2.1.1) \quad \mathcal{H} := \mathcal{L}_3^{-1} \otimes (\text{MC}_{\chi^{-1}}(\mathcal{L}_3 \otimes (\text{MC}_\chi(\text{MC}_{\chi^{-1}}(\mathcal{L}_1) \otimes \text{MC}_{\chi^{-1}}(\mathcal{L}_2))))).$$

By Remark 1.2.2, The monodromy tuple of  $\mathcal{G}$  is given by

$$(2.1.2) \quad \mathbf{h} = (h_1, h_2, h_3, h_4) = \mathbf{g}_3^{-1} \otimes (\text{MC}_{\zeta_3^{-1}}(\mathbf{g}_3 \otimes (\text{MC}_{\zeta_3}(\text{MC}_{\zeta_3^{-1}}(\mathbf{g}_1) \otimes \text{MC}_{\zeta_3^{-1}}(\mathbf{g}_2))))),$$

where  $\mathbf{g}_3^{-1} = (1, 1, \zeta_3^{-1}, \zeta_3)$  and where we have used the convention of Section 1.2 for the tensor product of tuples. Using the explicit receipt for the computation of  $\text{MC}_\lambda$ , one can evaluate Equation (2.1.2) and finds matrices for the monodromy tuple  $\mathbf{h}$  (these are given in the Appendix).

**2.2 Proof of Theorem 1:** The following result gives a proof of Theorem 1, (i) and (ii) (see Introduction):

**2.2.1 Proposition.** *Let*

$$\rho = \pi_1(\mathbb{P}^1 \setminus \{x_1, x_2, x_3\}) \longrightarrow \text{GL}_7(\mathbb{C})$$

*be the monodromy representation of  $\mathcal{H}$ . Then the following holds*

- *The Zariski closure of the image of  $\rho$  coincides with  $G_2(\mathbb{C})$ .*
- *The representation  $\rho$  (resp. the local system  $\mathcal{H}$ ) is  $G_2$ -rigid.*
- *The Jordan canonical forms of the local monodromies at  $x_1, x_2, x_3, \infty$  are as follows (respectively):*

$$\text{J}(2, 2, 1, 1, 1), \text{J}(2, 2, 1, 1, 1), \text{diag}(1, \zeta_3, \zeta_3, \zeta_3, \zeta_3^{-1}, \zeta_3^{-1}, \zeta_3^{-1}), \text{J}(3, 3, 1).$$

**Proof:** The claim on the Jordan canonical forms is obvious from the matrices given in the Appendix. Since the tuples  $\text{MC}_{\zeta_3^{-1}}(\mathbf{g}_i)$ ,  $i = 1, 2$ , are contained in the group  $\text{Sp}_2(\mathbb{C}) = \text{SL}_2(\mathbb{C})$ , their Kronecker product  $\text{MC}_{\zeta_3^{-1}}(\mathbf{g}_1) \otimes \text{MC}_{\zeta_3^{-1}}(\mathbf{g}_2)$  is contained in the orthogonal group  $\text{SO}_4(\mathbb{C})^4 \leq \text{GL}_4(\mathbb{C})^4$ . Moreover, the elements of  $\text{MC}_{\zeta_3^{-1}}(\mathbf{g}_1) \otimes \text{MC}_{\zeta_3^{-1}}(\mathbf{g}_2)$  can easily be seen to generate an irreducible subgroup of  $\text{GL}_4(\mathbb{C})$ . Thus, by [5], Cor. 3.6, or by [8], Thm. 2.9.8, the local system  $\mathcal{H}$  is irreducible (i.e., the elements of  $\mathbf{h}$  generate an irreducible subgroup of  $\text{GL}_7(\mathbb{C})$ ). Since the elements of  $\text{MC}_{\zeta_3^{-1}}(\mathbf{g}_1) \otimes \text{MC}_{\zeta_3^{-1}}(\mathbf{g}_2)$  are contained in  $\text{SO}_4(\mathbb{C})$ , it follows from [5], Cor. 5.15, that the elements of  $\mathbf{h}$  are contained in the group  $\text{SO}_7(\mathbb{C})$ . By an elementary computation (using the computer algebra system MAGMA [9]), one can check that the matrices stabilize a one-dimensional subspace of the third exterior power  $\Lambda^3(\mathbb{C}^7)$ . Thus, by the results of [1], the image of  $\rho$  is contained in  $G_2(\mathbb{C})$ . By the classification of bireflection groups given in [7], Thm. 7.1 and Thm. 8.3, the Zariski closure of the image of  $\rho$  can be seen to coincide (up to  $\text{GL}_7(\mathbb{C})$ -conjugation) with  $G_2(\mathbb{C})$ . By Prop. 1.1.1, the structure of the Jordan canonical forms of  $h_1, \dots, h_4$  then implies that the representation is  $G_2$ -rigid.  $\square$

**2.2.2 Remark.** (i) By taking 6-th roots of unity instead of third roots of unity for the definition of  $\mathbf{g}_1$  and  $\mathbf{g}_2$ , and by taking

$$\mathbf{s} := \text{MC}_{\zeta_6^{-1}}(\mathbf{g}_1) \otimes \text{MC}_{\zeta_6^{-1}}(\mathbf{g}_2),$$

one obtains another orthogonally rigid quadruple in  $\text{SO}_4(\mathbb{C})$ , whose Jordan canonical forms coincide with those of  $\text{MC}_{\zeta_3^{-1}}(\mathbf{g}_1) \otimes \text{MC}_{\zeta_3^{-1}}(\mathbf{g}_2)$ . After application of  $\mathbf{g}_3^{-1} \otimes \text{MC}_{\zeta_3^{-1}} \circ \mathbf{g}_3 \otimes \text{MC}_{\zeta_3}$  one obtains an orthogonally rigid tuple  $\mathbf{h}'$  in the group  $\text{SO}_7(\mathbb{C})$  which has the same tuple of Jordan canonical forms as the above tuple  $\mathbf{h} \in G_2(\mathbb{C})^4$ . (Thus the containment of the elements of  $\mathbf{h}$  in the group  $G_2$  cannot be derived from the information on the Jordan canonical forms alone.)

(ii) Proof of Theorem 1 (iii): Using the tensor structure of the group  $\text{SO}_4(\mathbb{C})$ , it can be checked that any irreducible and orthogonally rigid tuple in the group  $\text{SO}_4(\mathbb{C})$  is  $\text{GL}_4(\mathbb{C})$ -conjugate either to the above tuple  $\mathbf{s}$  or to  $\text{MC}_{\zeta_3^{-1}}(\mathbf{g}_1) \otimes \text{MC}_{\zeta_3^{-1}}(\mathbf{g}_2)$ . It follows then from the invertibility of  $\text{MC}_\lambda$  (see [5], Thm. 5.3, or [8], 5.1.5) that any tuple in the group  $\text{SO}_7(\mathbb{C})$  which has the same tuple of Jordan canonical forms is either  $\text{GL}_7(\mathbb{C})$ -conjugate to  $\mathbf{h}$  or to  $\mathbf{h}'$ .

**2.2.3 Proposition.** (Proof of Theorem 1 (iii) and (iv))

- (i) *The local system  $\mathcal{H}$  is  $G_2$ -rigid and  $\text{SO}_7$ -rigid but not  $\text{GL}_7$ -rigid.*
- (ii) *The rigidity conjecture holds for the  $G_2$ -rigid local system  $\mathcal{H}$ .*

**Proof:** The first claim follows from Theorem [11] considering the Jordan canonical forms of the monodromy tuple of  $\mathcal{H}$ , given in Prop. 2.2.1. The second claim follows by an iterative application of the Künneth-formula and Prop. 1.2.3.  $\square$

**2.2.4 Remark.** (i) Let  $3 < q$  be a natural number and  $\zeta_q$  be a primitive  $q$ -th root of unity. Let  $\mathbf{g}_1$  be as above and let  $\mathbf{g}'_2 = (1, \zeta_q \zeta_3, \zeta_q^{-1} \zeta_3, \zeta_3)$ . Let further  $\mathbf{g}_{q,1} = (1, \zeta_q^{-1}, 1, \zeta_q)$ ,  $\mathbf{g}_{q,2} = (1, \zeta_q^{-1}, \zeta_q, 1)$ ,  $\mathbf{g}_{q,3} = (1, 1, \zeta_q^{-1} \zeta_3, \zeta_3^{-1} \zeta_q)$  and  $\mathbf{g}_{q,4} = (1, 1, \zeta_3^{-1}, \zeta_3)$ . Using the sequence

$$\begin{aligned} & \mathbf{g}_{q,4} \otimes \text{MC}_{\zeta_q \zeta_3}(\mathbf{g}_{q,3} \otimes (\text{MC}_{\zeta_3^{-1}}(\mathbf{g}_{q,2} \otimes (\text{MC}_{\zeta_q^{-1}}(\mathbf{g}_{q,1} \otimes \\ & (\text{MC}_{\zeta_3^{-1}}(\mathbf{g}_1) \otimes (\mathbf{g}_{q,1} \otimes \text{MC}_{\zeta_q \zeta_3^{-1}}(\mathbf{g}'_2)))))) \end{aligned}$$

one obtains tuples  $\mathbf{h}_q$  whose tuples of Jordan canonical forms are

$$(\text{J}(2, 2, 1, 1, 1), \text{J}(2, 2, 1, 1, 1), \text{diag}(1, \zeta_3, \zeta_3, \zeta_3, \zeta_3^{-1}, \zeta_3^{-1}, \zeta_3^{-1})),$$



$$\text{diag}(\zeta_q, \zeta_q, 1, 1, 1, \zeta_q^{-1}, \zeta_q^{-1}).$$

The authors expect these tuples to have similar properties as the above considered tuple  $\mathbf{h}$  but lack of computational power prevents a proof of these statements in the general case.

- (ii) Assume for simplicity that  $\{x_1, x_2, x_3, \infty\} = \{0, \pm 1, \infty\}$  and let

$$f : \mathbb{P}^1 \setminus \{0, \pm 1, \infty\} \rightarrow \mathbb{P}^1 \setminus \{0, 1, \infty\}, x \mapsto x^2.$$

One can show that the local system  $\mathcal{H}$  is the pullback  $\mathcal{H} = f^*\mathcal{F}$ , where  $\mathcal{F}$  is the  $G_2$ -rigid local system associated to a triple  $(f_1, f_2, f_3) \in G_2(\mathbb{C})^3$  whose Jordan canonical forms are  $\text{diag}(1, -\zeta_3, -\zeta_3, \zeta_3, -\zeta_3^{-1}, -\zeta_3^{-1}, \zeta_3^{-1})$ ,  $J(2, 2, 1, 1, 1)$ , and an element having one Jordan block of length one and eigenvalue  $-1$ , one Jordan block of length three and eigenvalue  $-1$ , and one Jordan block of length three and eigenvalue  $1$  (here,  $f_1$  gives the monodromy at  $0$ ,  $f_2$  gives the monodromy at  $1$ , and  $f_3$  gives the monodromy at  $\infty$ ).

### 3 Appendix: The explicit monodromy tuple of $\mathcal{H}$

The monodromy tuple of the local system  $\mathcal{H} \in \text{LS}(\mathbb{P}^1 \setminus \{x_1, x_2, x_3\})$  is given by  $(h_1, \dots, h_4)$ , where  $h_4 = (h_1 h_2 h_3)^{-1}$  and  $h_1, h_2, h_3 \in \text{GL}_7(\mathbb{C})$  are as follows:

$$h_1 = \begin{pmatrix} 1 & -3 & \zeta_3 - 1 & 0 & \zeta_3 - 4 & 0 & 2\zeta_3 + 4 \\ 0 & 3\zeta_3 + 1 & 2\zeta_3 + 1 & 0 & 2\zeta_3 + 1 & -2\zeta_3 - 1 & 0 \\ 0 & -3\zeta_3 & -2\zeta_3 & 0 & -2\zeta_3 - 1 & 2\zeta_3 + 1 & 0 \\ 0 & 3\zeta_3 + 3 & \zeta_3 + 2 & 1 & \zeta_3 + 2 & -\zeta_3 - 2 & 0 \\ 0 & 3\zeta_3 + 6 & 3 & 0 & 4 & -3 & 0 \\ 0 & 3\zeta_3 + 3 & \zeta_3 + 2 & 0 & \zeta_3 + 2 & -\zeta_3 - 1 & 0 \\ 0 & 6 & -2\zeta_3 + 2 & 0 & -2\zeta_3 + 2 & 2\zeta_3 - 2 & 1 \end{pmatrix}$$

$$h_2 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \zeta_3 - 1 & 1 & 0 & 2\zeta_3 + 1 & 0 & 0 & 0 \\ 3 & 0 & 1 & -2\zeta_3 - 1 & -3 & 0 & 2\zeta_3 + 4 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$h_3 = \begin{pmatrix} \zeta_3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \zeta_3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \zeta_3 & 0 & 0 & 0 & 0 \\ \zeta_3 + 2 & 0 & 0 & -\zeta_3 - 1 & 0 & 0 & 0 \\ 0 & \zeta_3 + 2 & 0 & 0 & -\zeta_3 - 1 & 0 & 0 \\ 0 & 3\zeta_3 + 3 & \zeta_3 + 2 & 0 & 0 & -\zeta_3 - 1 & 0 \\ 0 & 0 & 0 & 0 & \zeta_3 - 1 & 0 & 1 \end{pmatrix}.$$

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